

Fixed Point Theorem of Expanding Mapping without Continuity in Cone Metric Space over Banach Algebra

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ABSTRACT: In this paper, we present fixed point theorems of expansion mapping by using c-sequence on cone metric space over Banach algebra without the assumption of normality.

KEYWORDS: Cone Metric space over Banach algebra, Expansion mapping, c-sequence, Fixed point theorems, Spectral radius.

I. INTRODUCTION

In 2007, Huang et al[1] generalized the concept of a metric space by considering vector valued-metrics with values in an ordered real Banach Space and introduced cone metric space; the set \mathbb{R} of real numbers was replaced by an ordered Banach space E . They defined the convergence via interior points of the cone by which the order in Banach space E is defined. They proved some fixed point theorems of contractive mappings. After that, many authors established the results of contractive mappings in cone metric space. But there are very few results about expanding mappings in the literature.

Recently, in 2013 H. Liu et al [2] announced the notion of cone metric space over Banach algebra by replacing Banach space by Banach algebra and obtained some fixed point theorems of generalized Lipschitz mapping by replacing the constant k with spectral radius $r(k)$ which is weaker condition.

In this paper we obtained fixed point theorem of expanding mapping on cone metric space over Banach algebra.

II. PRELIMINARIES

Let A be a real Banach algebra. i.e. A is a real Banach space in which an operation of multiplication is defined subject to the following properties, for all $x, y, z \in A, \alpha \in \mathbb{R}$

1. $(xy)z = x(yz)$
2. $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$;
3. $\alpha(xy) = (\alpha x)y = x(\alpha y)$;
4. $\|xy\| \leq \|x\|\|y\|$

In this paper we assume A is a Banach algebra with a unity (i.e. a multiplicative identity) e such that $ex = xe = x$ for all $x \in A$. An elements $x \in A$ is said to be invertible if there is an inverse element $y \in A$, such that $yx = xy = e$. The inverse of x is denoted by x^{-1} . [3]

Proposition 2.1:[3] Let A be a Banach algebra with a unit e , and $x \in A$. If the spectral radius $r(x)$ of x is less than 1 i.e.,

$$r(x) = \lim_{n \rightarrow \infty} \|x^n\|^{\frac{1}{n}} = \inf_{n \geq 1} \|x^n\|^{\frac{1}{n}} < 1.$$

then $e - x$ is invertible. Actually

$$(e - x)^{-1} = \sum_{i=0}^{\infty} x^i.$$

Now recall the concepts of cone for a Banach algebra A . A subset P of A is called a cone of A if

1. P is non-empty closed and $\{e, \theta\} \subset P$;
2. $\alpha P + \beta P \subset P$ for all non-negative real numbers α, β ;
3. $P^2 = PP \subset P$;
4. $P \cap (-P) = \{\theta\}$,

Where θ denotes the null of the Banach algebra A . for a given cone $P \subset A$, we can define a partial ordering \leq with respect to P by $x \leq y$ if and only if $y - x \in P$. $x < y$ will stand for $x \leq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int}P$, $\text{int}P$ denotes the interior of P . The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in A$, $0 \leq x \leq y \Rightarrow \|x\| \leq K\|y\|$

The least positive number satisfying above is called the normal constant of P . [1]

Definition 2.2: ([2]) Let A be a Banach algebra. Let X be a nonempty set. Suppose that a mapping $d: X \times X \rightarrow A$ satisfies the following conditions:

1. $0 \leq d(x, y)$ for all $x, y \in X$, and $d(x, y) = 0$ iff $x = y$.
2. $d(x, y) = d(y, x)$ for all $x, y \in X$
3. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space over Banach algebra A .

Example 2.3: ([2]) Let $A = M_n(\mathbb{R}) = a = \{(a_{ij})_{n \times n} \mid a_{ij} \in \mathbb{R}, \text{ for all } 1 \leq i, j \leq n\}$ be the algebra of all n -square real matrices, and define the norm

$$\|a\| = \sum_{k1 \leq i, j \leq n} |a_{ij}|$$

Then A is a real Banach algebra with the unit e the identity matrix.

Let $P = \{a \in A \mid a_{ij} \geq 0, \text{ for all } 1 \leq i, j \leq n\}$. Then $P \subset A$ is a normal cone with normal constant $K = 1$.

Let $X = M_n(\mathbb{R})$, and define the metric $d: X \times X \rightarrow A$ by

$$d(x, y) = d\left(\left(x_{ij}\right)_{n \times n}, \left(y_{ij}\right)_{n \times n}\right) = \left(|x_{ij} - y_{ij}|\right)_{n \times n} \in A.$$

Then (X, d) is a cone metric space over Banach algebra A with normality.

Example 2.4: ([2]) Let $A = C_{\mathbb{R}}^1[0, 1]$ and define a norm on A by $\|x\| = \|x\|_{\infty} = \|\dot{x}\|_{\infty}$ for $x \in A$. Define multiplication in A as just pointwise multiplication. Then A is a real unital Banach algebra with unit $e = 1$.

The set $P = \{x \in A: x \geq 0\}$ is a cone in A . Moreover, P is not normal.

Let $X = \{1, 2, 3\}$. Define $d: X \times X \rightarrow A$ by $d(1, 2)(t) = d(2, 1)(t) = d(2, 3)(t) = d(3, 2)(t) = e^t$,

$d(1, 3)(t) = d(3, 1)(t) = 2e^t, d(x, x)(t) = 0$. We see that (X, d) is a cone metric space over Banach algebra A without normality.

Definition 2.5: ([1, 2]) Let (X, d) be a cone metric space over Banach algebra A . Let $\{x_n\}$ be a sequence in X , then:

1. $\{x_n\}$ converges to x whenever for each $c \in A$ with $\theta \ll c$ there is a natural number N such that $d(x_n, x) \ll c$, for all $n \geq N$. We denote this by $\lim_{n \rightarrow \infty} x_n = x$ or $x_n \rightarrow x$ ($n \rightarrow \infty$).
2. $\{x_n\}$ is a Cauchy sequence whenever for each $c \in A$ with $\theta \ll c$ there is a natural number N such that $d(x_n, x_m) \ll c$, for all $n, m \geq N$.
3. (X, d) is a complete cone metric space, if every Cauchy sequence in X is convergent in X .

Lemma 2.6: ([4]) If E is a real Banach space with a solid cone P and if $\theta \leq u \ll c$ for each $\theta \ll c$ then $\theta = \theta$.

Lemma 2.7: ([4]) If E is a real Banach space with a solid cone P and if $\|x_n\| \rightarrow 0$ ($n \rightarrow \infty$), then for any $\theta \ll c$ there exists $N \in \mathbb{N}$ such that, for any $n > N$, we have $x_n \ll c$.

Remark 2.8: ([5]) If $r(k) < 1$ then $\|k^n\| \rightarrow 0$ ($n \rightarrow \infty$).

Lemma 2.9: ([6]) The limit of a convergent sequence in a cone metric space is unique.

III. MAIN RESULTS

Definition 3.1: ([7]) Let P be a solid cone in a Banach space A . A sequence $\{x_n\} \subset P$ is a c -sequence if for each $c \gg \theta$ there exists $n_0 \in \mathbb{N}$ such that $x_n \ll c$ for $n \geq n_0$.

Proposition 3.2: ([7]) Let P be a solid cone in a Banach space A . A sequence $\{x_n\}$ and $\{y_n\}$ be a sequences in P . If $\{x_n\}$ and $\{y_n\}$ are c -sequences and $\alpha, \beta > 0$, then $\{\alpha x_n + \beta y_n\}$ is a c -sequence.

Proposition 3.3: ([7]) Let P be a solid cone in Banach algebra A . A sequence $\{x_n\}$ be a sequence in P . Then the following conditions are equivalent:

- (1) $\{x_n\}$ a c -sequence.
- (2) For each $c \gg \theta$ there exists $n_0 \in \mathbb{N}$ such that $x_n < c$ for $n \geq n_0$.
- (3) For each $c \gg \theta$ there exists $n_1 \in \mathbb{N}$ such that $x_n \ll c$ for $n \geq n_1$.

Proposition 3.4: ([5]) Let P be a solid cone in a Banach algebra A . A sequence $\{x_n\}$ be a sequence in P . Suppose that $k \in P$ is an arbitrarily given vector and $\{x_n\}$ is a c -sequence in P . Then $\{kx_n\}$ is a c -sequence.

Proposition 3.5: ([5]) Let A be a Banach algebra A with unit e , P be a cone in A and \leq be semi-order generated by the cone P . The following assertion hold true.

- (1) For any $x, y \in A, a \in P$ with $x \leq y$, we have $ax \leq ay$.
- (2) For any sequence $\{x_n\}, \{y_n\} \subset A$ with $x_n \rightarrow x (n \rightarrow \infty)$ and $y_n \rightarrow y (n \rightarrow \infty)$ where $x, y \in A$, we have $x_n y_n \rightarrow xy (n \rightarrow \infty)$.

Proposition 3.6: ([5]) Let (X, d) be complete cone metric space over a Banach algebra A and let P be the underlying solid cone in Banach algebra A . Let $\{x_n\}$ be a sequence in X . If $\{x_n\}$ converges to $x \in X$, then we have:

- (1) $\{d(x_n, x)\}$ is a c -sequence.
- (2) For any $p \in \mathbb{N}$, $\{d(x_n, x_{n+p})\}$ is a c -sequence.

Now we define generalized expanding mappings.

Definition 3.7: Let (X, d) be a cone metric space over Banach algebra A and $T : X \rightarrow X$. Then T is called a generalized expanding mapping, if for every $x, y \in X$ there exists $k \in P$ with $r(k) > 1$ such that $d(Tx, Ty) \geq kd(x, y)$.

Now we shall prove some fixed point theorems of generalized expanding mapping in the setting of cone metric space with Banach algebra.

Lemma 3.8: ([3]) Let A be a Banach algebra, if x is invertible, $\lambda \in \sigma(x)$ if and only if $\lambda^{-1} \in \sigma(x^{-1})$.

Lemma 3.9: Let A be a Banach algebra A with unity e , P be a cone in A if x is invertible such that $r(x) > 1$ then $r(x^{-1}) < 1$

Proof: By definition of spectral radius

$$r(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\}$$

Let $r(x) > 1$.

i.e. $r(x) = \sup\{|\lambda| : \lambda \in \sigma(x)\} > 1$

i.e. $|\lambda| > 1$

Therefore, $|\lambda|^{-1} < 1$

So $|\lambda^{-1}| < 1$

So $\sup\{|\lambda^{-1}| : \lambda^{-1} \in \sigma(x^{-1})\} < 1$

i.e. $r(x^{-1}) < 1$. ■

Theorem 3.10: Let (X, d) be a complete cone metric space over Banach algebra A and let the mapping $T : X \rightarrow X$ be onto and satisfies the condition

International Journal of Innovative Research in Science, Engineering and Technology

An ISO 3297: 2007 Certified Organization

Volume 6, Special Issue 1, January 2017

International Conference on Recent Trends in Engineering and Science (ICRTES 2017)

20th-21st January 2017

Organized by

Research Development Cell, Government College of Engineering, Jalagon (M. S), India

$$d(Tx, Ty) \geq k d(x, y) \tag{1}$$

for all $x, y \in X$, and $k \in P, k$ is invertible where $r(k) > 1$. Then, T has a fixed point in X .

Proof: Since $T: X \rightarrow X$ is onto mapping so for each $x_0 \in X$, there exists $x_1 \in X$ such that $Tx_1 = x_0$

Similarly, for each $n \geq 1$ there exists $x_{n+1} \in X$ such that $x_n = Tx_{n+1}$.

If $x_{n-1} = x_n$ then x_n is a fixed point of T .

Thus, Suppose that $x_{n-1} \neq x_n$ for all $n \geq 1$.

Then

$$\begin{aligned} d(x_n, x_{n-1}) &= d(Tx_{n+1}, Tx_n) \\ &\geq k(d(x_{n+1}, x_n)) \\ &= kd(x_n, x_{n+1}) \\ d(x_n, x_{n+1}) &\leq k^{-1}d(x_n, x_{n-1}) \\ &= hd(x_{n-1}, x_n), \quad \text{where } h = k^{-1} \end{aligned}$$

Since $r(k) > 1$ so by lemma 3.9 $r(h) = r(k^{-1}) < 1$.

Form this we get $d(x_n, x_{n+1}) \leq h^n d(x_0, x_1)$ where $h = k^{-1}$ and

Now for $n < m$ we have by triangle inequality

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \dots + d(x_{m-1}, x_m) \\ &\leq (h^n + h^{n+1} + \dots + h^{m-1}) d(x_0, x_1) \\ &\leq h^n (e + h + h^2 + h^3 + \dots + h^{m-n-1}) d(x_0, x_1) \\ &\leq h^n (e - h)^{-1} d(x_0, x_1). \end{aligned}$$

By Remark 2.8 and lemma 2.7 it follows that, for $c \in A$ with $\theta \ll c$, there exists $N \in \mathbb{N}$ such that, for any $m > n > N$, we have

$$d(x_n, x_m) \leq h^n (e - h)^{-1} d(x_0, x_1) \ll c.$$

Which implies that $\{x_n\}$ is a Cauchy sequence.

Since X is complete, there exists $q \in X$ such that $Tx_{n+1} = x_n \rightarrow q (n \rightarrow \infty)$,

Consequently, we can find a $r \in X$ such that $Tr = q$. Now we show that $r = q$

Substituting $x = r, y = x_{n+1}$ in (1), we get

$$d(q, x_n) = d(Tr, Tx_{n+1}) > kd(r, x_{n+1})$$

By Triangle inequality, we have

$$\begin{aligned} d(q, x_n) &\leq d(q, x_{n+1}) + d(x_{n+1}, x_n) \\ d(q, x_{n+1}) + d(x_{n+1}, x_n) &\geq kd(r, x_{n+1}) \\ d(r, x_{n+1}) &\leq k^{-1}(d(q, x_{n+1}) + d(x_{n+1}, x_n)) \\ d(r, x_{n+1}) &\leq h(d(q, x_{n+1}) + d(x_{n+1}, x_n)) \end{aligned}$$

From the definition 3.1 Proposition 3.2, 3.3, 3.4 and 3.6 we have $d(r, x_{n+1}) \leq y_n$ where y_n is a c-sequence in cone P .

Hence, for each $c \gg \theta$ we have $d(r, x_{n+1}) \ll c$, so $d(r, x_{n+1}) = \theta$

So $x_{n+1} \rightarrow r$ From lemma 2.9 we have $r = q$. ■

REFERENCES

- [1] Huang L.G., X. Zhang, "Cone metric spaces and fixed point theorems of contractive mappings," J. Mat. Anal. Appl. **332** (2) 1468–1476, (2007).
- [2] Liu H., Xu S., "Cone metric spaces with Banach algebras and fixed point theorems of generalized Lipschitz mappings", Fixed Point Theory Appl. **2013**,320 (2013).
- [3] Rudin, W., "Functional Analysis", 2ndedn, McGraw Hill, New York(1991).
- [4] Radenovic S., Rhoades B.E., "Fixed point theorem for two non-self mappings in cone metric spaces", Comput. Math. Appl. **57**,1701-1707(2009).
- [5] Radenovic S., Xu S., "Fixed point theorems of generalized Lipschitz mappings on cone metric spaces over Banach algebra without assumption of normality", Fixed Point Theory Appl. **2014**,102,(2014).
- [6] Zhang X., "Common fixed point Theorems of Lipschitz type mappings in cone metric spaces", Acta Math. Sin. **53** (6), 1139-1148 (2010) (in Chinese)
- [7] Kadelburg Z., Radenovic S., "A note on various types of cones and fixed point results in cone metric spaces", Asian J. Math. Appl. **2013** Article ID ama0104 (2013).